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General solutions of the Wess-Zumino consistency condition for the Weyl anomalies

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Abstract: The general solutions of the Wess-Zumino consistency condition for the Weyl anomalies are derived in a purely algebraic manner. The solutions are obtained, in arbitrary dimensions, by explicitly computing the cohomology of the corresponding Becchi-Rouet-Stora-Tyutin differential in the space of integrated local functions at ghost number unity.

Keywords: Field Theories in Higher Dimensions, Anomalies in Field and String Theories, BRST Symmetry.

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Contents

1. Introduction

Although the Weyl (or conformal, or trace) anomalies were discovered about 30 years ago [1, 2], it is only very recently that their general structure was displayed in a purely algebraic manner, in arbitrary dimensions and independently of any regularization scheme [3].

The central equations that determine the candidate anomalies in quantum field theory are the Wess-Zumino (WZ) consistency conditions [4]. As is well-known, the determination of the general solution of the WZ consistency conditions boils down to the computation of the cohomology of the corresponding Becchi-Rouet-Stora-Tyutin (BRST) differential [5] in the space of local functionals with ghost number one. We refer to the book [6] for a pedagogical review and many references on the subject of anomalies in quantum field theory, while the works [7, 8] contain and review the most general results for Einstein-Yang-Mills and Yang-Mills gauge theories, in the presence of antifields. The literature on Weyl anomalies is huge. A very nonexhaustive list of references can be found, e.g., in [9 − 14].

The cohomological formulation for the determination of the Weyl anomalies was initiated in the pioneering works [15, 16], with results up to spacetime dimension $n = 6$ and the general structure guessed for arbitrary (even) n. The authors of these works found that the Weyl anomalies comprise (i) the integral over spacetime of σ, the Weyl parameter, times the Euler density of the manifold, plus (ii) terms that are given by (the integral of) σ times strictly Weyl-invariant scalar densities. Some of the terms from (ii) can be trivially obtained from contractions of products of the conformally invariant Weyl tensor, while the others are more complicated and involve covariant derivatives of the Riemann tensor.

These important cohomological results in dimensions $n = 4$ and $n = 6$ were obtained by listing all the possible terms on the basis of dimensionality and diffeomorphism invariance and by inserting them into the WZ consistency condition. The structure of the four-dimensional conformal anomalies was rederived later [17, 18], using the WZ conditions. Still, no systematic pattern emerged for the general structure of the Weyl anomalies in arbitrary dimension n.

Such results appeared later, in [19]. By applying dimensional regularization on the effective gravitational action generated by a conformally invariant matter system, the authors of [19] could confirm the structure found in [16] and extended the results to arbitrary n. The Euler term from class (i) was called “type-A Weyl anomaly”, while the terms of (ii) were called “type-B anomalies”. Very interestingly, from the structure of the poles in the effective action, it was observed [19] that the type-A anomaly appeared in a similar way to the non-Abelian chiral anomaly in Yang-Mills gauge theory. That the type-A anomaly should arise via some “descent identity” was therefore suggested. Subsequently, this suggestion was taken as a work hypothesis in [20].

More recently, in the holographic context of the anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence where the computation of the Weyl anomaly plays an important rôle [21 – 23], some cohomological considerations have been applied [24, 25] that confirm the structure found in [16, 19] and highlight the similarities between the type-A anomalies and the non-Abelian chiral anomalies.

From these considerations, it appeared that a purely algebraic understanding of the general structure of the Weyl anomalies, in arbitrary dimensions n and independently of the AdS/CFT correspondence or of any regularization scheme, was indeed most desirable and needed. Moreover, one could wonder whether descent equations in the manner of Stora and Zumino [26, 27] could appear in the classification of Weyl anomalies. It is the purpose of the present paper to answer these questions, providing detailed proofs. A brief report of some of our results was given in [3].

More precisely, following the antifield-independent approach as in [6] and using the powerful cohomological tools reviewed in [8], we solve the WZ consistency condition for the Weyl anomalies in arbitrary dimensions n. We demonstrate that the type-A anomaly is the *unique* solution associated with a non-trivial descent, whereas the type-B anomalies are given by trivial descents and can be computed by using the systematic, algebraic method of [28, 29]. We do not resort to dimensional analysis and that the spacetime dimension n must be even derives from consistency, it is not an assumption. These results are essentially obtained along the cohomological lines of [30 – 32] and crucially rely on preliminary results given in [29]. They imply the uniqueness of the known conformal anomalies and solve a question posed in [19] concerning the similitudes between the type-A anomaly and the non-Abelian chiral anomalies in Yang-Mills theories.

Incidentally, note also that our results provide a purely algebraic proof of the conjecture of differential geometry studied recently in [33].[1] This is yet another instance of the rich interplay between the study of anomalies in theoretical physics and mathematics.

2. Cohomological setting

In a theory that is classically diffeomorphism and Weyl invariant, the associated BRST differential is $s = s_D + s_W$, where s_D is the BRST differential corresponding to the diffeomorphisms and s_W corresponds to the Weyl transformations. As in [16], we consider the

[1]H. Baum is thanked for having pointed out these works to us.

purely gravitational part of the cohomological problem, where the spacetime metric $g_{\mu\nu}$ is an external classical field. Apart from the (invertible) metric, the other fields are the diffeomorphisms ghosts ξ^μ and the Weyl ghost ω, with ghost number $gh(\xi^\mu) = gh(\omega) = 1$. Spacetime indices are denoted by Greek letters and run over the values $0, 1, \ldots, n-1$. Flat, tangent space indices are denoted by Latin letters. The action of the BRST differential s on the fields $\Phi^A = \{g_{\mu\nu}, \xi^\mu, \omega\}$ is

$$
\begin{aligned}
s_D g_{\mu\nu} &= \xi^\rho \partial_\rho g_{\mu\nu} + \partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\rho g_{\mu\rho}\,, &(2.1)\\
s_W g_{\mu\nu} &= 2\,\omega\, g_{\mu\nu}\,, &(2.2)\\
s_D \xi^\mu &= \xi^\rho \partial_\rho \xi^\mu\,, &(2.3)\\
s_D \omega &= \xi^\rho \partial_\rho \omega\,, \quad s_W \xi^\mu = 0 = s_W \omega\,. &(2.4)
\end{aligned}
$$

The anomalies a_1^n are given by the solutions of the WZ consistency conditions

$$
s a_1^n + d\, b_2^{n-1} = 0\,, \qquad a_1^n \neq s p_0^n + d\, q_1^{n-1}\,, \tag{2.5}
$$

where superscripts denote the form degree whereas subscripts indicate the ghost number. All the cochains a_1^n, b_2^{n-1}, p_0^n and q_1^{n-1} are local forms and d is the total exterior derivative. A local p-form b^p depends on the fields Φ^A and their derivatives up to some finite (but otherwise unspecified) order, which is denoted by $b^p = \frac{1}{p!}\, dx^{\mu_1} \ldots dx^{\mu_p}\, b_{\mu_1 \ldots \mu_p}(x, [\Phi^A])$.

Since we are seeking Weyl anomalies, the ghost degree of a_1^n is carried entirely by (a derivative of) ω. Decomposing the WZ consistency conditions (2.5) with respect to the Weyl-ghost degree, one finds

$$
\begin{aligned}
s_D a_1^n + d\, b_2^{n-1} &= 0\,, &(2.6)\\
s_W a_1^n + d\, c_2^{n-1} &= 0\,, \quad a_1^n \neq s_W p_0^n + d\, f_1^{n-1} &(2.7)\\
\forall\, p_0^n \;\; s.t. \;\; s_D p_0^n + d\, h_1^{n-1} &= 0\,. &(2.8)
\end{aligned}
$$

In words, we have to compute the cohomology $H^{1,n}(s_W|d)$ of the Weyl BRST differential s_W modulo total derivatives, in the space of diffeomorphism-invariant local n-forms. As a matter of fact, an important result of [16] is that it is always possible, by adding a local Bardeen-Zumino counterterm to the action, to shift away the pure diffeomorphism part of the candidate anomaly a_1^n, leaving only the pure Weyl part of a_1^n. This is consistent with the fact that it is always possible to ensure diffeomorphism invariance throughout the process of regularization, at the price of losing Weyl invariance upon quantization. Actually, this can be taken as a definition of the Weyl anomaly.

Before attacking the problem (2.6)–(2.8), it is useful to reformulate the equations for the computation of $H^{1,n}(s|d)$ in slightly different terms. One can perform the Stora trick which consists in uniting the differentials $s = s_D + s_W$ and d into a single differential $\tilde{s} = s + d$. Then, the WZ consistency condition (2.5) and its descent are encapsulated in

$$
\tilde{s}\,\alpha = 0\,, \quad \alpha \neq \tilde{s}\,\zeta + constant \tag{2.9}
$$

for the local total forms α and ζ of total degrees $G = n+1$ and $G = n$. Local total forms are by definition formal sums of local forms with different form degrees and ghost numbers,

$\alpha = \sum_{p=0}^{n} a^{p}_{G-p}$, the total degree being simply the sum of the form degree and the ghost number. As proved in [30], the cohomology of s in the space of local functionals (integrals of local n-forms) and at ghost number g is locally isomorphic to the cohomology of $\tilde{s}$ in the space of local total forms at total degree $G = g + n$. Furthermore, the cohomological problem can be restricted, locally, to the $\tilde{s}$-cohomology on local total forms belonging to a subspace $\mathcal{W}$ of the space of local total forms [30]:

$$\begin{aligned} \tilde{s}\,\alpha(\mathcal{W}) &= 0\,, & \alpha(\mathcal{W}) &\neq \tilde{s}\,\zeta(\mathcal{W}) + constant\,, \\ totdeg(\alpha) &= n+g\,, & totdeg(\zeta) &= n+g-1\,. \end{aligned} \tag{2.10}$$

The subspace $\mathcal{W}$, closed under the action of $\tilde{s}$, is given by local total forms depending on so-called tensor fields $\{\mathcal{T}^i\}$ at total degree zero and on so-called generalized connections $\{\widetilde{C}^N\}$ at total degree unity. The latter decompose into a part with ghost number one and form degree zero plus a part having ghost number zero but form degree unity: $\widetilde{C}^N = \widehat{C}^N + \mathcal{A}^N$. For a purely gravitational theory in metric formulation, invariant under diffeomorphisms and Weyl transformations, the space $\mathcal{W}$ was found in [29].

The solution of the problem (2.9) will thus have the form

$$\alpha(\mathcal{W}) = \widetilde{C}^{N_1} \ldots \widetilde{C}^{N_n} \widetilde{C}^{N_{n+1}}\, a_{N_1 \ldots N_{n+1}}(\mathcal{T})$$

where the anomalies are given (up to an unessential constant coefficient) by the top form-degree component of the local total form $\alpha(\mathcal{W})$:

$$a^n_1 = \mathcal{A}^{N_1} \ldots \mathcal{A}^{N_n} \widehat{C}^{N_{n+1}}\, a_{N_1 \ldots N_{n+1}}(\mathcal{T})\,.$$

Now, we are ready to attack the system (2.6)—(2.8). This is done by solving (2.10) at total degree $G = n+1$ with $\tilde{s}$ replaced by $\tilde{s}_W = s_W + d$ and taking the equations (2.6), (2.8) into account. These last two equations tell us that cocycles and coboundaries of $\tilde{s}_W$ must be diffeomorphism-invariant. It is important to specify the space in which one computes the anomaly. Without any restriction of this kind, we would have the triviality of all the Weyl anomaly candidates $a^n_1 = \omega f(\mathcal{T}) d^n x$ where $f(\mathcal{T})$ is a Weyl-invariant scalar density. Indeed, $\omega f(\mathcal{T}) d^n x = \tilde{s}_W [f(\mathcal{T}) d^n x \frac{1}{n}\, ln(\sqrt{-g})]$. However, the local form $p^n_0 = \frac{1}{n}\, ln(\sqrt{-g}) f(\mathcal{T}) d^n x$ is forbidden because it fails to obey the condition (2.8).

3. Solution of the Wess-Zumino consistency condition

To reiterate, we must look for $\tilde{s}_D$-invariant $(n+1)$-local total forms $\alpha(\mathcal{W})$ satisfying

$$\tilde{s}_W \alpha(\mathcal{W}) = 0\,, \quad \alpha(\mathcal{W}) \neq \tilde{s}_W \zeta(\mathcal{W}) + constant\,, \tag{3.1}$$

where $\zeta(\mathcal{W})$ must be $\tilde{s}_D$-invariant. The solution will take the general form

$$\alpha(\mathcal{W}) = 2\omega\, \tilde{C}^{N_1} \ldots \tilde{C}^{N_n}\, a_{N_1 \ldots N_n}(\mathcal{T})\,. \tag{3.2}$$

Before continuing with the solution of the WZ consistency condition for the Weyl anomalies, we must spend some time in order to explain the various symbols that appear in the above

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equation (3.2). In the same process, we will display the gauge covariant algebra associated with the BRST transformations (2.1)–(2.4) and relate it to the conformal algebra $\mathfrak{so}(n,2)$, in the flat space limit.

The space $\mathcal{T}$ of tensor fields is generated by the (invertible) metric $g_{\mu\nu}$ together with the so-called W-tensors $\{W_{\Omega_i}\}$, $i \in \mathbb{N}$ [29]. It is only necessary to recall here that the W-tensors are tensors under general coordinate transformations and transform under s_W according to $s_W W_{\Omega_i} = \omega_\alpha \mathbf{\Gamma}^\alpha W_{\Omega_i}$, where $\omega_\alpha = \partial_\alpha \omega$ and the n generators $\mathbf{\Gamma}^\alpha$ $(0 \leqslant \alpha \leqslant n-1)$ act only on the W-tensors. These tensors are built recursively with the help of the formula $W_{\Omega_k} = (\nabla_{\alpha_k} + K_{\beta\alpha_k}\mathbf{\Gamma}^\beta) W_{\Omega_{k-1}} = \mathcal{D}_{\alpha_k} W_{\Omega_{k-1}}$, where $K_{\alpha\beta} = \frac{1}{n-2}\left(\mathcal{R}_{\alpha\beta} - \frac{1}{2(n-1)}\, g_{\alpha\beta}\mathcal{R}\right)$ and $W_{\Omega_0} = W^\mu{}_{\nu\rho\sigma}$ is the conformally invariant Weyl tensor. The symbol ∇ denotes the usual torsion-free metric-compatible covariant differential associated with the Christoffel symbols $\Gamma^\mu{}_{\nu\rho}$, while $\mathcal{R}_{\alpha\beta} = R^\mu{}_{\alpha\mu\beta}$ is the Ricci tensor with $R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu{}_{\nu\sigma} + \cdots$ the Riemann tensor. The scalar curvature is given by $\mathcal{R} = g^{\alpha\beta}\mathcal{R}_{\alpha\beta}$.

The Weyl tensor can be written as

$$W^\mu{}_{\nu\rho\sigma} = R^\mu{}_{\nu\rho\sigma} - 2\left(\delta^\mu_{[\rho} K_{\sigma]\nu} - g_{\nu[\rho} K_{\sigma]}{}^\mu\right), \tag{3.3}$$

where curved (square) brackets denote strength-one complete (anti)symmetrization.

The following notation is useful and explains the meaning of the superindices Ω_i, $i \in \mathbb{N}$:

$$\begin{aligned} W_{\Omega_0} &= W^\mu{}_{\nu\rho\sigma}\,, \quad W_{\Omega_1} = \mathcal{D}_{\alpha_1} W_{\Omega_0} = \mathcal{D}_{\alpha_1} W^\mu{}_{\nu\rho\sigma}\,, \quad \ldots \\ W_{\Omega_k} &= \mathcal{D}_{\alpha_k} W_{\Omega_{k-1}} = \mathcal{D}_{\alpha_k}\mathcal{D}_{\alpha_{k-1}} \ldots \mathcal{D}_{\alpha_2}\mathcal{D}_{\alpha_1} W^\mu{}_{\nu\rho\sigma}\,, \end{aligned}$$

where $\mathcal{D}$ is the Weyl-covariant derivative as introduced[2] in [29].

In the latter work we introduced and operator that counts the number of metric tensors appearing in a given expression. An inverse metric brings a minus-one contribution. Explicitly, $\Delta_g^{\rm ex} = g_{\mu\nu}\frac{\partial}{\partial g_{\mu\nu}}$. For example, $\Delta_g^{\rm ex}(g_{\alpha\beta}g^{\gamma\delta}) = 0$ and $\Delta_g^{\rm ex}(g^{\gamma\sigma}g^{\lambda\nu}W_{\Omega_k}) = -2(g^{\gamma\sigma}g^{\lambda\nu}W_{\Omega_k})$. By definition, the operator $\Delta_g^{\rm ex}$ gives zero when applied on the W-tensors $\{W_{\Omega_i}\,,\ i \in \mathbb{N}\}$ and on the generalized connections $\{\tilde{C}^N\}$. Then, denoting[3] by $\Delta^\mu{}_\nu$ the generators of $GL(n)$-transformations of world indices acting on a type-$(1,1)$ tensor T^β_α as $\Delta^\mu{}_\nu T^\beta_\alpha = \delta^\mu_\alpha T^\beta_\nu - \delta^\beta_\nu T^\mu_\alpha$, the gauge covariant algebra $\mathcal{G}$ generated by $\{\Delta_N\} = \{\Delta_g^{\rm ex}, \mathcal{D}_\nu, \Delta^\mu{}_\nu, \mathbf{\Gamma}^\alpha\}$ reads [29]

$$[\Delta^\mu{}_\nu, \mathbf{\Gamma}^\alpha] = -\delta^\alpha_\nu \mathbf{\Gamma}^\mu\,, \quad [\Delta^\mu{}_\nu, \mathcal{D}_\alpha] = \delta^\mu_\alpha \mathcal{D}_\nu\,, \tag{3.4}$$

$$[\Delta^\rho{}_\mu, \Delta^\sigma{}_\nu] = \delta^\rho_\nu \Delta^\sigma{}_\mu - \delta^\sigma_\mu \Delta^\rho{}_\nu\,, \quad [\mathbf{\Gamma}^\alpha, \mathbf{\Gamma}^\beta] = 0\,, \tag{3.5}$$

$$[\mathcal{D}_\beta, \mathbf{\Gamma}^\alpha] = \mathcal{P}^{\nu\alpha}_{\beta\mu}\Delta^\mu{}_\nu - \delta^\alpha_\beta \Delta_g^{\rm ex}\,, \tag{3.6}$$

$$[\mathcal{D}_\rho, \mathcal{D}_\sigma] = -W^\mu{}_{\nu\rho\sigma}\Delta^\nu{}_\mu - C_{\alpha\rho\sigma}\,\mathbf{\Gamma}^\alpha\,, \tag{3.7}$$

where $C_{\alpha\mu\nu} = 2\,\nabla_{[\nu}K_{\mu]\alpha}$ is the Cotton tensor and $\mathcal{P}^{\nu\alpha}_{\beta\mu} = (-g^{\nu\alpha}g_{\beta\mu} + \delta^\nu_\beta\delta^\alpha_\mu + \delta^\alpha_\beta\delta^\nu_\mu)$. The operator $\Delta_g^{\rm ex}$ commutes with all the other generators. As shown in [29], the gauge covariant

[2] V. Wünsch informed us that such a construction had been obtained previously, see e.g. [34] and references therein. Similar constructions and other references can be found in [35]. Apparently, all those works lead back to the ones of T. Y. Thomas [36].

[3] Notation is slightly changed as compared with [29]. In passing, we also correct a couple of typos present therein.

algebra $\mathcal{G}$ is realized on the space $\mathcal{W}$ of tensor fields $\mathcal{T}$ and generalized connections $\{\tilde{C}^N\}$. The second term on the right-hand side of (3.6) was not written in [29]. However, it must be present in order for the commutation relation $[\mathcal{D}_\beta, \mathbf{\Gamma}^\alpha]$ to be realized on the metric tensor as well, recalling $\mathbf{\Gamma}^\alpha g_{\mu\nu} = 0 = \mathcal{D}_\rho g_{\mu\nu}$.

The generalized connections $\{\tilde{C}^N\}$ present in (3.2) are obtained from [29], setting the diffeomorphisms ghosts ξ^μ to zero. All of them are Grassmann-odd and read

$$\begin{aligned}
\{\tilde{C}^N\} &= \{2\omega\,, dx^\nu\,, \tilde{C}^\nu{}_\mu\,, \tilde{\omega}_\alpha\}\,,\\
\tilde{C}^\nu{}_\mu &= \Gamma^\nu{}_{\mu\rho}\, dx^\rho\,, \quad \tilde{\omega}_\alpha = \omega_\alpha - K_{\alpha\rho}\, dx^\rho\,, \quad \omega_\alpha = \partial_\alpha \omega\,.
\end{aligned}$$

Then, with $\{\Delta_N\} = \{\Delta_g^{\mathrm{ex}}\,, \mathcal{D}_\nu\,, \Delta^\mu{}_\nu\,, \mathbf{\Gamma}^\alpha\}$, the action of $\tilde{s}_W$ on the tensor fields $\{\mathcal{T}^i\}$ and generalized connections $\{\tilde{C}^N\}$ can be written in the very concise form

$$\tilde{s}_W \mathcal{T}^i = \tilde{C}^N \Delta_N \mathcal{T}^i\,, \quad \tilde{s}_W \tilde{C}^N = \tfrac{1}{2}\, \tilde{C}^L \tilde{C}^K \mathcal{F}_{KL}{}^N(\mathcal{T})\,,$$

where $\mathcal{F}_{KL}{}^N(\mathcal{T})$ denote the structure functions of the gauge covariant algebra $\mathcal{G}$:

$$[\Delta_M, \Delta_N] = \mathcal{F}_{MN}{}^L(\mathcal{T}) \Delta_L\,.$$

The relation $\tilde{s}_W \tilde{C}^N = \frac{1}{2}\, \tilde{C}^L \tilde{C}^K \mathcal{F}_{KL}{}^N(\mathcal{T})$ generalizes the so-called "Russian formula". It is rather remarkable that the sole equations (2.1)–(2.4) completely determine the gauge covariant algebra (3.4)–(3.7).

A relevant issue concerning the algebra $\mathcal{G}$ given by (3.4)–(3.7) (it is *not* a Lie algebra) is whether it can be related to the conformal algebra $\mathfrak{so}(n,2)$. After all, we are considering a general class of theories that are classically diffeomorphism and Weyl invariant, and we know that such theories, in the flat limit $g_{\mu\nu} \to \eta_{\mu\nu}$, reduce to conformally-invariant theories. Introducing the new set of generators $\{\, P_\mu\,,\; K_\nu\,,\; M_{\mu\nu}\,,\; D \,\}$ via

$$\begin{aligned}
\{\, \Delta_{\mu\nu}\,,\; \mathbf{\Gamma}_\alpha\,,\; D \,\} &= \{\, g_{\mu\rho} \Delta^\rho{}_\nu\,,\; g_{\alpha\beta} \mathbf{\Gamma}^\beta\,,\; \delta^\mu_\nu \Delta^\nu{}_\mu - \Delta_g^{\mathrm{ex}} \,\}\,,\\
\{\, P_\mu\,,\; K_\nu\,,\; M_{\mu\nu} \,\} &= \left\{ \frac{1}{4}\, \mathcal{D}_\mu\,,\; 2\,\mathbf{\Gamma}_\nu\,,\; -2\,\Delta_{[\mu\nu]} \right\},
\end{aligned}$$

one gets from (3.4)–(3.7) the following gauge algebra:

$$\begin{aligned}
[P_\alpha, M_{\mu\nu}] &= 2\, g_{\alpha[\mu} P_{\nu]}\,, \qquad & [K_\alpha, M_{\mu\nu}] &= 2\, g_{\alpha[\mu} K_{\nu]}\,,\\
[D, P_\mu] &= P_\mu\,, & [D, K_\mu] &= -K_\mu\,,\\
[M_{\alpha\mu}, M_{\beta\nu}] &= 2\, g_{\alpha[\beta} M_{\nu]\mu} - 2\, g_{\mu[\beta} M_{\nu]\alpha}\,, & &\\
[P_\mu, K_\nu] &= 2\,(g_{\mu\nu} D + M_{\mu\nu})\,, & [K_\mu, K_\nu] &= 0\,,\\
[P_\mu, P_\nu] &= -\frac{1}{2}\, W^{\rho\sigma}{}_{\mu\nu}\, M_{\rho\sigma} - \frac{1}{2}\, C_{\alpha\mu\nu}\, K^\alpha & &
\end{aligned}$$

which is isomorphic to the conformal algebra $\mathfrak{so}(n,2)$ when $g_{\mu\nu} = \eta_{\mu\nu}$, as was to be expected. Discussions and references on soft algebras, soft group manifolds and the transition from curved to flat spacetime in this context can be found in [37].

After this short comment on the relation between the soft (gauge) covariant algebra $\mathcal{G}$ and the (rigid) conformal algebra $\mathfrak{so}(n,2)$, we can proceed with the solution of the WZ

	$\tilde{s}_W^{\mathrm{lie}}$	$\tilde{s}_W^{0}$	$\tilde{s}_W^{-1}$
$\tilde{C}^\nu{}_\mu$	$-\tilde{C}^\nu{}_\alpha\tilde{C}^\alpha{}_\mu$	0	$\tilde{s}_W^{-1}\tilde{C}^\nu{}_\mu$
$\tilde{\omega}_\alpha$	$\tilde{C}^\beta{}_\alpha\tilde{\omega}_\beta$	$\frac{1}{2}\,dx^\rho dx^\sigma C_{\alpha\rho\sigma}$	0
ω	0	$dx^\mu\tilde{\omega}_\mu$	0
$g_{\mu\nu}$	$\tilde{C}^\beta{}_\alpha\Delta^\alpha{}_\beta g_{\mu\nu}$	$2\,\omega\,g_{\mu\nu}$	0
W_{Ω_i}	$\tilde{C}^\beta{}_\alpha\Delta^\alpha{}_\beta W_{\Omega_i}$	$dx^\mu\mathcal{D}_\mu W_{\Omega_i}+\tilde{\omega}_\alpha\mathbf{\Gamma}^\alpha W_{\Omega_i}$	0

Table 1: Decomposition of the action of $\tilde{s}_W$

consistency condition for the Weyl anomaly and its schematic solution (3.2). Because of the fermionic nature of the Weyl ghost ω, the generalized connections $\tilde{C}^{N_i}$ in (3.2) must all be different from 2ω, otherwise $\alpha(\mathcal{W})$ vanishes. The appearance of the undifferentiated Weyl ghost ω in (3.2) is not an assumption. The Weyl-ghost dependence of the anomaly a_1^n can entirely be expressed in terms of the undifferentiated ghost ω, by integrating by parts: $\sqrt{-g}\;\omega_\alpha V^\alpha \;=\; \partial_\alpha(\omega\sqrt{-g}\;V^\alpha) - \omega\sqrt{-g}\;\nabla_\alpha V^\alpha$. We can now proceed with (3.1) and expand $\alpha(\mathcal{W})$ in powers of the connection $\tilde{C}^\nu{}_\mu$,

$$\alpha(\mathcal{W}) = \sum_{k=0}^{m}\alpha(\mathcal{W})\,,\quad N_C\,\alpha_k = k\,\alpha_k\,,\quad N_C = \tilde{C}^\nu{}_\mu\frac{\partial^L}{\partial\tilde{C}^\nu{}_\mu}\,.$$

On $\mathcal{W}$, the differential $\tilde{s}_W$ decomposes into three parts,

$$\tilde{s}_W\,\alpha(\mathcal{W}) = (\tilde{s}_W^{\mathrm{lie}}+\tilde{s}_W^{0}+\tilde{s}_W^{-1})\,\alpha(\mathcal{W}) \tag{3.8}$$

which have N_C-degrees 1, 0, -1 respectively.

The action of $\tilde{s}_W^{\mathrm{lie}}$, $\tilde{s}_W^{0}$ and $\tilde{s}_W^{-1}$ can be summarized in table 1, together with

$$\tilde{s}_W^{-1}\tilde{C}^\nu{}_\mu = \frac{1}{2}\,dx^\rho dx^\sigma W^\nu{}_{\mu\rho\sigma} + \mathcal{P}^{\nu\alpha}_{\beta\mu}\,\tilde{\omega}_\alpha\,dx^\beta\,.$$

The cocycle condition $\tilde{s}_W\alpha = 0$ thus decomposes into

$$0 = \tilde{s}_W^{\mathrm{lie}}\alpha_m \tag{3.9}$$

$$0 = \tilde{s}_W^{0}\alpha_m + \tilde{s}_W^{\mathrm{lie}}\alpha_{m-1} \tag{3.10}$$

$$0 = \tilde{s}_W^{-1}\alpha_m + \tilde{s}_W^{0}\alpha_{m-1} + \tilde{s}_W^{\mathrm{lie}}\alpha_{m-2}$$

$$\vdots$$

In the first equation, a contribution of the form $\tilde{s}_W^{\mathrm{lie}}\beta_{m-1}$ can be redefined away by subtracting the trivial piece $\tilde{s}_W\beta_{m-1}$ from α. The solution of equation (3.9) is known because we know the Lie algebra cohomology of $\mathfrak{gl}(n)$. Indeed, $\mathfrak{gl}(n)\cong\mathbb{R}\oplus\mathfrak{sl}(n)$ is reductive. Since all the fields of $\mathcal{W}$ transform according to finite-dimensional linear representations of $\mathfrak{gl}(n)$, we have

$$\alpha_m = \varphi_i(dx,\omega,\tilde{\omega}_\alpha,\mathcal{T})P^i(\tilde{\theta})\,,\quad \tilde{s}_W^{\mathrm{lie}}\varphi_i = 0\,. \tag{3.11}$$

The $P^i(\tilde{\theta})$ are linearly independent polynomials in the primitive elements $\tilde{\theta}_K$ of the Lie algebra cohomology of $\mathfrak{gl}(n)$. The $\tilde{\theta}_K$'s are monomials in the $\tilde{C}^\nu{}_\mu$'s and correspond to the independent Casimir operators of $\mathfrak{gl}(n)$.

Inserting (3.11) in (3.10) gives

$$(\tilde{s}^0_W \varphi_i) P^i(\tilde{\theta}) + \tilde{s}^{\text{lie}}_W \alpha_{m-1} = 0\,.$$

Again, using the Lie algebra cohomology, we deduce

$$\tilde{s}^0_W \varphi_i(dx, \omega, \tilde{\omega}_\alpha, \mathcal{T}) = 0 \quad \forall\, i\,. \tag{3.12}$$

We can assume that none of the φ_i's is of the form $\tilde{s}_W \vartheta(dx, \omega, \tilde{\omega}_\alpha, \mathcal{T})$ because otherwise we could remove that particular φ_i by subtracting the trivial piece $\tilde{s}_W(\vartheta P^i)$ from α. Such a subtraction does not clash with the other redefinitions made so far. In particular it does not reintroduce a term $\tilde{s}^{\text{lie}}_W \beta_{m-1}$ in (3.11) because of the definition of the P^i's.

Hence, since the φ_i's do not depend on the $\tilde{C}^\nu{}_\mu$'s, we see that they are determined by the $\tilde{s}_W$-cohomology in the space of $\mathfrak{gl}(n)$-invariant local total forms $\varphi(dx, \omega, \tilde{\omega}_\alpha, \mathcal{T})$. [The coboundary condition $\varphi(dx, \omega, \tilde{\omega}_\alpha, \mathcal{T}) = \tilde{s}_W \vartheta(dx, \omega, \tilde{\omega}_\alpha, \mathcal{T})$ requires ϑ to be $\mathfrak{gl}(n)$-invariant, by expanding the equation in $\tilde{C}^\nu{}_\mu$.] We thus have to solve

$$\tilde{s}_W \varphi(dx, \omega, \tilde{\omega}_\alpha, \mathcal{T}) = 0\,, \qquad \varphi(dx, \omega, \tilde{\omega}_\alpha, \mathcal{T}) \neq \tilde{s}_W \vartheta(dx, \omega, \tilde{\omega}_\alpha, \mathcal{T})\,, \tag{3.13}$$

$$\tilde{s}^{\text{lie}}_W \varphi = 0 = \tilde{s}^{\text{lie}}_W \vartheta\,. \tag{3.14}$$

In order to solve the above equations, we decompose the relation $\tilde{s}_W \varphi(dx, \omega, \tilde{\omega}_\alpha, \mathcal{T}) = 0$ into parts with definite degree in the appropriately symmetrized W-tensor fields (see [29]) and analyze it starting from the part with lowest degree. The decomposition is unique and thus well-defined thanks to the algebraic independence of the appropriately symmetrized W-tensors. The decomposition of $\tilde{s}_W$ takes the form $\tilde{s}_W = \sum_{k \geqslant 0} \tilde{s}^{(k)}_W$, $[N_W, \tilde{s}^{(k)}_W] = k\, \tilde{s}^{(k)}_W$ where N_W is the counting operator for the — appropriately symmetrized — W-tensors.

The $\mathfrak{gl}(n)$-invariant local total form $\varphi(dx, \omega, \tilde{\omega}_\alpha, \mathcal{T})$ decomposes into a sum of $\mathfrak{gl}(n)$-invariant terms

$$\begin{aligned} \varphi(dx, \omega, \tilde{\omega}_\alpha, \mathcal{T}) &= \varphi_{(0)}(dx, \omega, \tilde{\omega}_\alpha, g_{\mu\nu}) + \sum_{k>0} \varphi_{(k)}(dx, \omega, \tilde{\omega}_\alpha, \mathcal{T})\,, \\ N_W\, \varphi_{(k)} &= k\, \varphi_{(k)}\,. \end{aligned}$$

The condition $\tilde{s}_W \varphi = 0$ requires, at lowest order in the tensor fields,

$$\tilde{s}^{(0)}_W \varphi_{(0)}(dx, \omega, \tilde{\omega}_\alpha, g_{\mu\nu}) = 0\,. \tag{3.15}$$

Furthermore, we can remove any piece of the form $\tilde{s}^{(0)}_W \vartheta_{(0)}(dx, \omega, \tilde{\omega}_\alpha, g_{\mu\nu})$ from $\varphi_{(0)}$ by subtracting the trivial piece $\tilde{s}_W \vartheta_{(0)}$ from φ. Hence, $\varphi_{(0)}$ is actually determined by the $\tilde{s}^{(0)}_W$-cohomology in the space of $\mathfrak{gl}(n)$-invariant local total forms with no dependence on the W-tensors. In particular, we can assume $\varphi_{(0)} \neq \tilde{s}^{(0)}_W \vartheta_{(0)}(dx, \omega, \tilde{\omega}_\alpha, g_{\mu\nu})$. Writing $\varphi_{(0)} = \omega\, \ell_{(0)}(dx, \tilde{\omega}_\alpha, g_{\mu\nu})$, the condition (3.15) translates into $dx^\mu \tilde{\omega}_\mu \ell_{(0)} = 0$. The most general $\ell_{(0)}(dx, \tilde{\omega}_\alpha, g_{\mu\nu})$ reads

$$\begin{aligned} \ell_{(0)}(dx, \tilde{\omega}_\alpha, g_{\mu\nu}) = &\sum_{p=0}^{n} \eta_p\, dx^{\alpha_1} \dots dx^{\alpha_p}\, \tilde{\omega}_{\alpha_1} \dots \tilde{\omega}_{\alpha_p} + \\ &+ \sum_{p=0}^{n} \frac{\lambda_p}{\sqrt{-g}}\, \varepsilon^{\nu_1 \dots \nu_p \mu_1 \dots \mu_{n-p}}\, g_{\mu_1 \alpha_1} \dots g_{\mu_{n-p} \alpha_{n-p}} dx^{\alpha_1} \dots dx^{\alpha_{n-p}}\, \tilde{\omega}_{\nu_1} \dots \tilde{\omega}_{\nu_p}\,, \end{aligned}$$

where η_p and λ_p are constants, $0 \leqslant p \leqslant n$. In the second line of the above equation, we have inserted an appropriate power of $\det(g_{\mu\nu})$ in order that the corresponding local total form φ possesses the correct weight to provide us with a candidate anomaly (the ε-symbol is the completely antisymmetric weight–1 Levi-Civita tensor density), as imposed by condition (2.6). The condition $dx^\mu \tilde{\omega}_\mu \ell_{(0)} = 0$ imposes $\eta_p = 0$, $0 \leqslant p \leqslant n-1$, which yields

$$\varphi_{(0)}(dx, \tilde{\omega}_\alpha, g_{\mu\nu}) = \eta_n \, \omega \; dx^{\alpha_1} \dots dx^{\alpha_n} \, \tilde{\omega}_{\alpha_1} \dots \tilde{\omega}_{\alpha_n} + \\ + \frac{\omega}{\sqrt{-g}} \sum_{p=0}^{n} \lambda_p \varepsilon^{\nu_1 \dots \nu_p \mu_1 \dots \mu_{n-p}} g_{\mu_1 \alpha_1} \dots g_{\mu_{n-p} \alpha_{n-p}} dx^{\alpha_1} \dots dx^{\alpha_{n-p}} \tilde{\omega}_{\nu_1} \dots \tilde{\omega}_{\nu_p}.$$

However, the first term is a local total form of degree $2n+1$, which is too much since we look for local total forms of degree $(n+1)$.[4] Accordingly, we set $\eta_n = 0$.

The next step consists in determining whether $\varphi_{(0)}$ is $\tilde{s}_W^{(0)}$-trivial or not. We find that all the terms in $\varphi_{(0)}$ are $\tilde{s}_W^{(0)}$-trivial, except one. Indeed,

$$\tilde{s}_W^{(0)} J = \omega \, \frac{(n-2p)}{\sqrt{-g}} \, \varepsilon^{\nu_1 \dots \nu_p \mu_1 \dots \mu_{n-p}} \, g_{\mu_1 \alpha_1} \dots g_{\mu_{n-p} \alpha_{n-p}} dx^{\alpha_1} \dots dx^{\alpha_{n-p}} \, \tilde{\omega}_{\nu_1} \dots \tilde{\omega}_{\nu_p} \,,$$
$$\text{where } \; J = \frac{1}{\sqrt{-g}} \, \varepsilon^{\nu_1 \dots \nu_p \mu_1 \dots \mu_{n-p}} \, g_{\mu_1 \alpha_1} \dots g_{\mu_{n-p} \alpha_{n-p}} dx^{\alpha_1} \dots dx^{\alpha_{n-p}} \, \tilde{\omega}_{\nu_1} \dots \tilde{\omega}_{\nu_p} \,,$$

so that only the term with $p = n/2$ survives in the $\tilde{s}_W^{(0)}$-cohomology, leaving us with an $(n+1)$-total form $\varphi_{(0)}$.

Summarizing, with $m = \frac{n}{2}$ we have (up to an irrelevant constant coefficient)

$$\varphi_{(0)} = \frac{\omega}{\sqrt{-g}} \, \varepsilon^{\nu_1 \dots \nu_m}{}_{\mu_1 \dots \mu_m} \, dx^{\mu_1} \dots dx^{\mu_m} \, \tilde{\omega}_{\nu_1} \dots \tilde{\omega}_{\nu_m}. \tag{3.16}$$

Of course, this term exists only in even dimensions.

We may now ask what is the completion $\varphi = \varphi_{(0)} + \sum_k \varphi_{(k)}$ of (3.16) that would be invariant under the full differential $\tilde{s}_W$. This question can be answered by using a decomposition of φ and $\tilde{s}_W$ with respect to the $\tilde{\omega}_\alpha$-degree. The differential $\tilde{s}_W$ decomposes into a part noted $\tilde{s}_\flat$ which lowers the $\tilde{\omega}_\alpha$-degree by one unit, a part noted $\tilde{s}_\natural$ which does not change the $\tilde{\omega}_\alpha$-degree and a part noted $\tilde{s}_\sharp$ which raises the $\tilde{\omega}_\alpha$-degree by one unit: $\tilde{s}_W = \tilde{s}_\flat + \tilde{s}_\natural + \tilde{s}_\sharp$. The action of these three parts of $\tilde{s}_W$ is given in table 2.

The decomposition of φ with respect to the $\tilde{\omega}_\alpha$-degree reads

$$\varphi = \Phi_m^{[m]} + \Phi_{m-1}^{[m+1]} + \dots + \Phi_1^{[n-1]} + \Phi_0^{[n]} \,,$$
$$\Phi_m^{[m]} = \varphi_{(0)} \,, \qquad m = \frac{n}{2} \,,$$

where each term $\Phi_r^{[n-r]}$ $(0 \leqslant r \leqslant m)$ is $\mathfrak{gl}(n)$-invariant, possesses a $\tilde{\omega}_\alpha$-degree r and explicitly contains the product of $(n-r)$ dx's. [Of course, some dx's are also hidden inside the $\tilde{\omega}_\alpha$'s.]

[4] At most, the corresponding factor $P(\tilde{\theta})$ being in this case $P(\tilde{\theta}) = 1$ and the Weyl anomaly thus reducing to $\alpha = \alpha_m = \varphi$, cf. (3.11).

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	$\tilde{s}_\flat$	$\tilde{s}_\natural$	$\tilde{s}_\sharp$
$\tilde{\omega}_\alpha$	$\frac{1}{2}\,dx^\rho dx^\sigma C_{\alpha\rho\sigma}$	$\tilde{C}^\beta{}_\alpha\tilde{\omega}_\beta$	0
ω	0	0	$dx^\mu\tilde{\omega}_\mu$
W_{Ω_i}	0	$\tilde{C}^\beta{}_\alpha\Delta^\alpha{}_\beta W_{\Omega_i} + dx^\mu\mathcal{D}_\mu W_{\Omega_i}$	$\tilde{\omega}_\alpha\mathbf{\Gamma}^\alpha W_{\Omega_i}$
$g_{\mu\nu}$	0	$\tilde{C}^\beta{}_\alpha\Delta^\alpha{}_\beta\, g_{\mu\nu} + 2\,\omega\, g_{\mu\nu}$	0
$\tilde{C}^\nu{}_\mu$	0	$-\tilde{C}^\nu{}_\alpha\tilde{C}^\alpha{}_\mu + \frac{1}{2}\,dx^\rho dx^\sigma W^\nu{}_{\mu\rho\sigma}$	$\mathcal{P}^{\nu\alpha}_{\beta\mu}\,\tilde{\omega}_\alpha\, dx^\beta$

Table 2: Action of $\tilde{s}_{W}$, decomposed w.r.t the $\tilde{\omega}_\alpha$-degree

Decomposing the cocycle condition $\tilde{s}_W\varphi = 0$ with respect to the $\tilde{\omega}_\alpha$-degree yields the following descent of equations

$$\begin{aligned}
\tilde{s}_\flat\Phi_1^{[n-1]} + \tilde{s}_\natural\Phi_0^{[n]} &= 0 \quad ,\\
\tilde{s}_\flat\Phi_2^{[n-2]} + \tilde{s}_\natural\Phi_1^{[n-1]} + \tilde{s}_\sharp\Phi_0^{[n]} &= 0 \quad ,\\
&\vdots\\
\tilde{s}_\flat\Phi_m^{[m]} + \tilde{s}_\natural\Phi_{m-1}^{[m+1]} + \tilde{s}_\sharp\Phi_{m-2}^{[m+2]} &= 0 \quad ,\\
\tilde{s}_\natural\Phi_m^{[m]} + \tilde{s}_\sharp\Phi_{m-1}^{[m+1]} &= 0 \quad ,\\
\tilde{s}_\sharp\Phi_m^{[m]} &= 0 \quad .
\end{aligned}$$

In the following theorem, we give the expression for $\Phi_r^{[n-r]}$, $0 \leqslant r \leqslant m$, such that $\varphi = \sum_{r=0}^m \Phi_r^{[n-r]}$ is a solution of $\tilde{s}_W\varphi = 0$ with $\Phi_m^{[m]} = \varphi_{(0)}$ (3.16). Furthermore, the n-form $\Phi_0^{[n]}$ is separately $\tilde{s}_W$-invariant and the top form degree component of φ is nothing but the type-A Weyl anomaly. The anomaly $\beta = \Phi_0^{[n]}$ gives rise to a trivial descent and is a linear combination of type-B anomalies obtained simply by contractions of products of Weyl tensors.

Theorem 1. *Let $\psi_{\mu_1\ldots\mu_{2p}}$ be the local total form*

$$\psi_{\mu_1\ldots\mu_{2p}} = \frac{\omega}{\sqrt{-g}}\,\varepsilon^{\alpha_1\ldots\alpha_r}{}_{\nu_1\ldots\nu_r\mu_1\ldots\mu_{2p}}\,\tilde{\omega}_{\alpha_1}\ldots\tilde{\omega}_{\alpha_r}\,dx^{\nu_1}\ldots dx^{\nu_r}\,,$$
$$p = m - r\,,\quad m = n/2\,,\quad 0 \leqslant r \leqslant m$$

and $W^{\mu\nu}$ the tensor-valued two-form

$$W^{\mu\nu} = W^\mu{}_\lambda\, g^{\lambda\nu} = \frac{1}{2}\,dx^\rho dx^\sigma W^\mu{}_{\lambda\rho\sigma}\, g^{\lambda\nu}\,.$$

Then, the local total forms $\Phi_r^{[n-r]}$ $(0 \leqslant r \leqslant m)$

$$\Phi_r^{[n-r]} = \frac{(-1)^p}{2^p}\,\frac{m!}{r!\,p!}\,\psi_{\mu_1\ldots\mu_{2p}}\,W^{\mu_1\mu_2}\ldots W^{\mu_{2p-1}\mu_{2p}}$$

obey the descent of equations

$$\begin{cases} \tilde{s}_\flat\Phi_r^{[n-r]} + \tilde{s}_\natural\Phi_{r-1}^{[n-r+1]} = 0 \quad , \\ \qquad\qquad \tilde{s}_\sharp\Phi_r^{[n-r]} = 0 \quad , \quad (1 \leqslant r \leqslant m) \end{cases}$$
$$\tilde{s}_\flat\Phi_1^{[n-1]} \;=\; 0 \;=\; \tilde{s}_W\,\Phi_0^{[n]}\,,$$

so that the following relations hold:

$$\tilde{s}_W \alpha = 0 = \tilde{s}_W \beta \,,$$
$$\alpha = \sum_{r=1}^{m} \Phi_r^{[n-r]} \,, \quad \beta = \Phi_0^{[n]} \,.$$

Proof. The proof follows by direct computation, using the tracelessness of the Weyl tensor and with the help of the identity $\nabla W^{\mu\nu} = 2\, C_\rho\, g^{\rho[\mu} dx^{\nu]}$ relating the covariant differential of the Weyl two-form $W^{\mu\nu}$ to the Cotton two-form $C_\rho = \frac{1}{2}\, dx^\mu dx^\nu\, C_{\rho\mu\nu}$.

Finally, we have the

Theorem 2. *(A) The top form-degree component a_1^n of α (cf. Theorem 1) satisfies the WZ consistency conditions for the Weyl anomalies. The WZ conditions for a_1^n give rise to a non-trivial descent and a_1^n is the* unique *anomaly with such a property, up to the addition of trivial terms and anomalies satisfying a trivial descent.*

(B) The anomaly $\beta = \Phi_0^{[n]}$ satisfies a trivial descent and is obtained by taking contractions of products of Weyl tensors (m of them in dimension $n = 2m$). The top form-degree component e_1^n of $(\alpha + \beta)$ is proportional to the Euler density of the manifold $\mathcal{M}_n$:

$$e_1^n \;=\; \frac{(-1)^m}{2^m}\, \omega\, (R_{a_1 b_1} \wedge \ldots \wedge R_{a_m b_m})\, \varepsilon^{a_1 b_1 \ldots a_m b_m} \,.$$

Proof. (A) When computing the solutions of (3.13) and (3.14), we used an expansion of $\varphi(dx, \omega, \tilde{\omega}_\alpha, \mathcal{T})$ in the number of (appropriately symmetrized) W-tensors and found a solution starting with a W-independent term $\varphi_{(0)}$ given in (3.16). This term, as we showed, gives rise to (a representative of) the so-called type-A anomaly. However, in order to compute the general solutions of (3.13) and (3.14), we must determine whether other solutions exist, that would start with a term $\varphi_{(\ell)}$ with $\ell > 0$. If one returns to the decomposition of local total forms in terms of form degree and ghost number, writing $\varphi(dx, \omega, \tilde{\omega}_\alpha, \mathcal{T}) = \sum_{r=1}^{q+1} b_r^{p-r+1}$, the problem (3.13), (3.14), takes on the usual descent-equation form

$$s_W\, b_1^p + d\, b_2^{p-1} = 0 \quad , \tag{3.17}$$
$$s_W\, b_2^{p-1} + d\, b_3^{p-2} = 0 \quad ,$$
$$\vdots$$
$$s_W\, b_q^{p-q+1} + d\, b_{q+1}^{p-q} = 0 \quad , \tag{3.18}$$
$$s_W\, b_{q+1}^{p-q} = 0 \quad (0 \leqslant q \leqslant p \leqslant n), \tag{3.19}$$

where every element b_{i+1}^{p-i} ($0 \leqslant i \leqslant q$) transforms as a local $(p-i)$-form under spacetime diffeomorphisms, so that $d\, b_{i+1}^{p-i} = \nabla b_{i+1}^{p-i}$ where $\nabla = dx^\mu \nabla_\mu$ is the Levi-Civita covariant differential. One assumes that the descent is displayed in its shortest expansion, i.e. that q is minimal. This means that b_{q+1}^{p-q} is non-trivial in $H^{q+1,p-q}(s_W|d)$ since otherwise $b_{q+1}^{p-q} = s_W\, \mu_q^{p-q} + d\, \mu_{q+1}^{p-q-1}$ and (3.18) would then become $s_W\, [b_q^{p-q+1} - d\mu_q^{p-q}] = 0$, which, upon redefining b_q^{p-q+1}, would imply that the descent has shortened by one step, contrary to the shortest-descent hypothesis.

A priori, the head of the descent, b_1^p, possesses a form degree $p \leqslant n$ because candidate anomalies are obtained by completing [see eqs. (3.9)–(3.11)] the product $\varphi(dx, \omega, \tilde{\omega}_\alpha, \mathcal{T})P(\tilde{\theta})$, where $P(\tilde{\theta})$ is a polynomial in the primitive elements $\tilde{\theta}_K$ of the Lie algebra cohomology of $\mathfrak{gl}(n)$ and possesses a non-vanishing form degree, except for the trivial element $P(\tilde{\theta}) = 1$. The ghost number of b_1^p must be one because the $P^i(\tilde{\theta})$'s have a vanishing Weyl-ghost degree. On the other hand, it is known that the condition (2.6), in the absence of (derivatives of) diffeomorphisms ghosts, admits only two kinds of terms [38]. The first have the general form $\mathcal{L}d^n x$ where the lagrangian density $\mathcal{L}$ is constructed out of the Riemann tensor, the matter fields, the Yang-Mills field strength and their covariant derivatives. The second class of terms contains the pure-gravity Chern-Simons densities that depend explicitly on the Riemannian connection one-form $\tilde{C}^\nu{}_\mu$ and on the undifferentiated curvature two-form $R^\mu{}_\nu = \frac{1}{2}\, R^\mu{}_{\nu\rho\sigma} dx^\rho dx^\sigma$. Since the candidate Weyl-anomalies are linear in the Weyl-ghost ω which plays the rôle of a matter field, we conclude that no Chern-Simons term can appear in a_1^n, and hence the only allowed polynomial $P(\tilde{\theta})$ is the trivial one, $P(\tilde{\theta}) = 1$, which in turn implies that one can set $p = n$ in the descent (3.17)–(3.19), without loss of generality.

The case where $q = 0$ means that the descent is trivial and the candidate anomalies satisfy $s_W a_1^n = 0$. These are the type-B Weyl anomalies that can be classified and computed systematically along the lines of [28, 29]. Accordingly, in what follows we assume $q > 0$.

The bottom of the descent is obtained from $\alpha(\mathcal{W})$ by taking its maximal $\tilde{\omega}_\alpha$-degree component and taking only the contribution ω_α of $\tilde{\omega}_\alpha = \omega_\alpha - dx^\mu K_{\mu\alpha}$. In other words, the bottom of the descent must not depend on the one-form potential $\mathcal{A}_\alpha = -dx^\mu K_{\mu\alpha}$. A priori, when determining the most general non-trivial bottom b_{q+1}^{n-q} in (3.19), the dependence on the space of W-tensors can be complicated. However, it was proved in [32] that, for any given (super) Lie algebra $\mathfrak{g}$, the solutions of non-trivial descents as in (3.17)–(3.19) can be computed, without loss of generality, in the small algebra $\mathcal{B}$ generated by the one-form potentials, the curvature two-forms, the ghosts and the exterior derivatives of the ghosts.

In the present setting, the curvature two-forms decompose into $W^\mu{}_\nu = \frac{1}{2}\, dx^\rho dx^\sigma\, W^\mu{}_{\nu\rho\sigma}$ and $C_\alpha = \frac{1}{2}\, dx^\rho dx^\sigma\, C_{\alpha\rho\sigma}$, which take their values along the generators $\Delta^\nu{}_\mu$ and $\mathbf{\Gamma}^\alpha$, respectively, as can be read off from (3.7). The algebra generated by $\{\Delta^\nu{}_\mu, \mathbf{\Gamma}^\alpha\}$ [see (3.4), (3.5)] is non-reductive, being isomorphic to the semi-direct sum of $\mathfrak{gl}(n)$ and the abelian translation-like algebra $\mathfrak{t}(n)$. In analogy with a Yang-Mills gauge theory, the rôle of the Killing metric is played here by $g_{\mu\nu}$ which obeys $\mathcal{D}_\rho g_{\mu\nu} = 0$. Another invariant object at our disposal is the Levi-Civita ε symbol. The exterior differentials of the ghosts give $dx^\alpha \omega_\alpha$ and $dx^\beta \partial_\beta \omega_\alpha$, but the latter must be rejected because they do not belong to $\mathcal{W}$.

To summarize, the bottom of the descent b_{q+1}^{n-q} can depend on the W-tensors only through the curvature two-forms C_α and $W^\mu{}_\nu$. It is linear in the undifferentiated ghost ω and must not depend on $\mathcal{A}_\alpha = -dx^\mu K_{\mu\alpha}$. Moreover, it is easy to see that the Cotton two-form C_α cannot enter b_{q+1}^{n-q} since otherwise, up to a trivial d-exact term, b_{q+1}^{n-q} would depend on $\mathcal{A}_\alpha$. This is because $C_{\alpha\mu\nu} = 2\, \nabla_{[\nu} K_{\mu]\alpha}$ and the fact that ∇ may be replaced by the exterior differential d inside the descent made of p-forms.

Hence, the general form of b^{n-q}_{q+1} is given by a linear combinaison of terms of the form $\omega\,\mathrm{Tr}\,(\prod_{i,j,k} W^{\mu_i}{}_{\nu_i}\,\omega_{\rho_j} dx^{\sigma_k})$ where the trace is obtained by using the metric and the ε symbol. The relation $W^{\mu}{}_{\nu}\omega_{\mu} = -s_{W} C_{\nu}$ (see e.g. [29]) shows that no $W^{\mu_i}{}_{\nu_i}$ can be contracted with a ω_{ρ}. Together with the identity $W^{\mu}{}_{\nu} dx^{\rho} g_{\rho\mu} = 0$, this shows that the indices of the $W^{\mu_i}{}_{\nu_i}$'s must be contracted among themselves.

Suppose first that we use no Levi-Civita ε symbol in order to contract the indices in $\prod_{j,k}\omega_{\rho_j} dx^{\sigma_k}$. The corresponding b^{n-q}_{q+1}'s look like $b^{n-q}_{q+1} \sim \omega\,\mathrm{Tr}\,(\prod_i W^{\mu_i}{}_{\nu_i})\prod_j^q \omega_{\rho_j} dx^{\rho_j}$. Taking the exterior derivative of such a term gives contributions where d hits ω and contributions when d hits one of the $W^{\mu_i}{}_{\nu_i}$'s. Trivially, $d(\omega_\alpha dx^\alpha) = 0$ because $\omega_\alpha = \partial_\alpha \omega$. Because in $d\,b^{n-q}_{q+1}$ one can replace $dW^{\mu}{}_{\nu}$ by $2\,C_\rho\, g^{\rho[\mu} dx^{\sigma]}\, g_{\sigma\nu}$ and because $W^{\mu}{}_{\nu} dx^{\rho} g_{\rho\mu} = 0$, only the contribution from $d\omega$ survives in $d\,b^{p-q}_{q+1}$. This provides terms of the form $d\,b^{p-q}_{q+1} \sim \mathrm{Tr}\,(\prod_i W^{\mu_i}{}_{\nu_i})\prod_j^{q+1}\omega_{\rho_j} dx^{\rho_j}$ that, in the space $\mathcal{Y}$ obtained from $\mathcal{W}$ by discarding the $\tilde{C}^{\mu}{}_{\nu}$'s, clearly belong to the cohomology of s_{W} — it suffices to use the results of [39], taking the linearized part of $d\,b^{p-q}_{q+1}$ — and therefore are obstructions to the lift (3.18) of b^{n-q}_{q+1}.

The only other possibilities in the expression of the candidate b^{n-q}_{q+1} are exhausted by

$$b^{n-q}_{q+1} \sim \omega\,\mathrm{Tr}\left(\prod_i W^{\mu_i}{}_{\nu_i}\right)\sqrt{-g}\;\varepsilon_{\sigma_1\dots\sigma_q\rho_1\dots\rho_{n-q}} g^{\sigma_1\tau_1}\dots g^{\sigma_q\tau_q}\,\omega_{\tau_1}\dots\omega_{\tau_q}\, dx^{\rho_1}\dots dx^{\rho_{n-q}}\,.$$

However, such terms are non-trivial in $H(s_{W},\mathcal{Y})$ iff $q = n/2$. Since the factor $\mathrm{Tr}\,(\prod_{i=1}^k W^{\mu_i}{}_{\nu_i})$ brings a form degree $2k$ and because the remaining factor in b^{n-q}_{q+1} already gives an n-form at the top of the descent, we conclude that $k=0$ and the bottom of the descent reduces to the only term ($m = n/2$)

$$b^{m}_{m+1} = \omega\,\sqrt{-g}\;\varepsilon_{\sigma_1\dots\sigma_m\rho_1\dots\rho_m}\, g^{\sigma_1\tau_1}\dots g^{\sigma_m\tau_m}\omega_{\tau_1}\dots\omega_{\tau_m}\, dx^{\rho_1}\dots dx^{\rho_m} \tag{3.20}$$

which is contained in (3.16). The latter term gives rise to the candidate anomaly α presented in Theorem 1. Because (3.20) is non-trivial in the cohomology $H(s_{W},\mathcal{Y})$, so is the corresponding a^n_1 in $H(s_{W}|d)$, taking into account (2.6) and (2.8). This proves part (A) of the theorem.

Part (B) is proved by direct computation. □

4. Conclusion

We solved the Wess-Zumino consistency condition for the Weyl anomalies by explicitly computing the cohomology of the corresponding BRST differential in the space of integrated local functions at ghost number unity. The analysis features descent equations à la Stora-Zumino and provides a general, purely algebraic understanding of the structure of the Weyl anomalies in arbitrary dimensions.

The approach followed here is purely cohomological and independent of any regularization scheme. No dimensional argument is used and the evenness of the spacetime dimension is a consequence of the Wess-Zumino consistency condition, as is the general structure of the Weyl anomalies.

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